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Note on the longitudinal vibration in a thin visco-elastic rod of variable cross-section,

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In this note, the longitudinal stress in a thin visco-elastic rod of variable cross-section has been obtained in the presence of a body force. The medium has been taken of the type considered by Achenbach & Chao (1962). The approximate result of the longitudinal stress has been obtained for small values of time. The convolution theorem and Laplace transform have been found suitable.

Introduction

In this paper, the longitudinal stress in a thin visco-elastic rod of variable cross-section, has been worked out in the presence of a body force in the medium of the type considered by Achenbach & Chao (1962). This stress-strain relation considered by them is given by

$$\left(D + \frac{m^{1/2}}{T}\right)^2 \sigma = E_1\left(D + \frac{m}{T}\right)^2 \epsilon, \qquad \ldots(1)$$

where D is the differential operator $\frac{d}{dt}$ and E_1, m, T are material constants σ and ϵ are the stress and strain, respectively. The body force has been considered as $\rho\, Ae^{-kx}\delta(t)$ where ρ is the density of the medium, A, the variable cross-section of the rod and $\delta(t)$, the Dirac delta function. Because of the complication in taking inverse transforms of $\bar{\sigma}$, approximate results of the longitudinal stress have been obtained for small values of time. The use of the convolution theorem and Laplace transform have been found suitable to solve the problem.

Equations

The stress-strain relation is taken in the form (1). The rod is supposed to be bounded by the sections $x = a$ and $x = b$, its axis being the $x -$ axis.

The equation of motion is given by

$$\frac{\partial}{\partial x}(A\sigma) + \rho A e^{-kx}.\,\delta(t) = \rho A\,\frac{\partial^2 u}{\partial t^2} \qquad \ldots(2)$$

where u is the displacement along the x-axis, of a section whose undisturbed position is at a distance x from the origin, A being the area of this section, ρ, the density of the medium. The body force is given by equation (2)

The strain is given by $\epsilon = \dfrac{\partial u}{\partial x}$ ...(3)

For linear variation of the cross-section, we write

$$A = A_0 x \qquad \text{.. (4)}$$

where A_0 is constant.

End conditions are assumed to be

$$\left.\begin{aligned} \sigma(x,t) &= -Pe^{-\omega_1 t} \text{ at } x = a \\ \text{and, } \quad &= 0 \quad \text{at } x = b \end{aligned}\right\} \qquad \text{.. (5)}$$

METHOD OF SOLUTION

Using (5), equation (3) becomes

$$\rho x e^{-kx}.\ \delta(t) + \sigma + x\frac{\partial \sigma}{\partial x} = \rho x \frac{\partial^2 u}{\partial t^2} \qquad \text{.. (6)}$$

Writing $\dfrac{m^{1/2}}{T} = R$, $\dfrac{m}{T} = S$ and using equations (1), (4), (7)

we get

$$\frac{\partial^4 \sigma}{\partial t^4} + 2R\frac{\partial^3 \sigma}{\partial t^3} + R^2\frac{\partial^2 \sigma}{\partial t^2} = \frac{E_1}{\rho}\left(\frac{\partial^4 \sigma}{\partial t^2 \partial x^2} + 2S\frac{\partial^3 \sigma}{\partial t\, \partial x^2} + S^2\frac{\partial^2 \sigma}{\partial x^2}\right)$$

$$+\frac{E_1}{\rho x}\left(\frac{\partial^3 \sigma}{\partial t^2 \partial x} + 2S\frac{\partial^2 \sigma}{\partial t\, \partial x} + S^2\frac{\partial \sigma}{\partial x}\right)$$

$$-\frac{E_1}{\rho x^2}\left(\frac{\partial^2 \sigma}{\partial t^2} + 2S\frac{\partial \sigma}{\partial t} + S^2 \sigma\right)$$

$$-E_1 k e^{-kx}\left\{\frac{\partial^2}{\partial t^2}\delta(t) + 2S\frac{\partial}{\partial t}\delta(t) + S^2\delta(t)\right\} \qquad \text{...(7)}$$

Applying the Laplace transform given by

$$\bar{f}(x,p) = \int_0^\infty e^{-pt}.\ f(x,t)dt$$

to equation (8), with the initial conditions,

$$\sigma(x, 0) = 0 \text{ and } \frac{\partial}{\partial t}\dot{\sigma}(x, 0) = 0, \; a \leqslant x \leqslant b. \qquad \ldots(8)$$

and writing $D \equiv \frac{\partial}{\partial x}, D^2 \equiv \frac{\partial^2}{\partial x^2}$, etc.

$$\text{we get } D^2\bar{\sigma} + \frac{1}{x}D\bar{\sigma} - \left\{\frac{1}{x^2} + q^2\right\}\bar{\sigma} = A_1 e^{-kx} \qquad \ldots(9)$$

$$\text{Where } A_1 = \rho k, \; c^2 = \frac{E_1}{\rho} \text{ and } q^2 = \frac{(p^2 + Rp)^2}{c^2(p + S)^2} \qquad (10)$$

The solution of equation (9) is given by (Inche 1956)

$$\bar{\sigma}_1(qx)v_1(x) + k_1(qx)v_2(x)$$

where $I_1(qx)$ and $k_1(qx)$ are two fundamental solutions of the reduced equation $D^2\bar{\sigma} + \frac{1}{x}D\bar{\sigma} - \left\{\frac{1}{x^2} + q^2\right\}\bar{\sigma} = 0$

and $v_1(x)$, $v_2(x)$ are given by

$$\left.\begin{aligned} v_1(x) &= V_1(x) + c_1 = -A_1 \int \frac{k_1(qx)}{\triangle\{I_1(qx), k_1(qx)\}} \cdot e^{-kx} dx + c_1 \\ v_2(x) &= V_2(x) + c_2 = A_1 \int \frac{I_1(qx)}{\triangle\{I_1(qx), k_1(qx)\}} \cdot e^{-kx} dx + c_2 \end{aligned}\right\} \qquad \ldots(11)$$

where $\triangle(I_1, k_1)$ is the Wronskian of $I_1(qx)$ and $k_1(qx)$ and c_1, c_2 are the integration constants. Hence,

$$\bar{\sigma} = c_1 I_1(qx) + c_2 k_1(qx) + I_1(qx)V_1(x)k_1(qx)V_2(x) \qquad \ldots(12)$$

Now, Laplace transform of (6) is given by

$$\left.\begin{aligned} \bar{\sigma} &= -\frac{P}{p + \omega_1} \quad \text{at } x = a \\ \text{and,} \quad &= 0 \quad \text{at } x = b \end{aligned}\right\} \qquad \ldots(13)$$

From equations (12) and (13) we get

$$c_1 = \frac{\begin{aligned}&[-k_1(qa)I_1(qb)V_1(b) - k_1(qa)k_1(qb)V_2(b) + k_1(qb). \\ &\qquad k_1(qa)V_2(a) + k_1(qa)I_1(qa)V_1(a) + \frac{P}{p+\omega_1} \cdot k_1(qb)]\end{aligned}}{I_1(qb)k_1(qa) - I_1(qa)k_1(qb)}$$

$$c_2 = \frac{\begin{aligned}&[-I_1(qb)I_1(qa)V_1(a) - I_1(qb)k_1(qa). \\ &\qquad V_2(a) + I_1(qa)k_1(qb)V_2(b) + I_1(qa)I_1(qb)V_1(b) - \frac{P}{p+\omega_1} \cdot I_1(qb)]\end{aligned}}{I_1(qb)k_1(qa) - I_1(qa)k_1(qb)} \qquad \ldots(14)$$

From (14) and (12) we observe that the form of $\bar{\sigma}$ is very complicated. Therefore, the approximate results for small values of time t have been evaluated.

We proceed as follows :

$$\triangle[I_1(qx), k_1(qx)] = \begin{vmatrix} I_1(qx), & k_1(qx) \\ I'_1(qx), & k'_1(qx) \end{vmatrix}, \text{ where the prime denotes}$$

differentiation with respect to x.

$$\cong \begin{vmatrix} \frac{1}{(2\pi q)^{1/2}}\left(\frac{e^{qx}}{x^{1/2}} - \frac{3\,e^{qx}}{8qx^{3/2}} + \cdots\right), & \left(\frac{\pi}{2q}\right)^{1/2}\left(\frac{e^{-qx}}{x^{1/2}} + \frac{3e^{-qx}}{8qx^{3/2}} - \cdots\right) \\ \frac{1}{(2\pi q)^{1/2}}\left(\frac{qe^{qx}}{x^{1/2}} - \frac{7}{8}\frac{e^{qx}}{x^{3/2}} + \cdots\right), & \left(\frac{\pi}{2q}\right)^{1/2}\left(-\frac{qe^{-qx}}{x^{1/2}} - \frac{7}{8}\frac{e^{-qx}}{x^{3/2}} - \cdots\right) \end{vmatrix}$$

$$\cong -\frac{1}{x}. \qquad \text{...(15)}$$

$$\text{Again } v_1(x) = V_1(x) + c_1 \cong A_1 \int \left(\frac{\pi}{2q}\right)^{1/2} \cdot \frac{x}{x^{1/2}} \cdot e^{-kx} \cdot e^{-qx} \cdot \left(1 + \frac{3}{8qx} + \cdots\right) dx + c_1$$

$$\cong -A_1\left(\frac{\pi}{2q}\right)^{1/2} \cdot \frac{e^{-(k+q)x}}{k+q} \cdot x^{1/2} + c_1 \qquad \text{...(16)}$$

$$v_2(x = V_2(x) + c_2 \cong -A_1 \int \frac{1}{(2\pi q)^{1/2}} \cdot \frac{e^{qx}}{x^{1/2}}\, x \cdot e^{-kx} \left(1 - \frac{3}{8qx} + \cdots\right) dx + c_2$$

$$\cong -A_1 \cdot \frac{1}{(2\pi q)^{1/2}} \cdot \frac{e^{(q-k)x}}{q-k} \cdot x^{1/2} + c_2 . \qquad \text{...(17)}$$

$$-k_1(qa)I_1(qb)V_1(b)I_1(qx) \cong \frac{A_1}{4q^2} \cdot \frac{1}{(x.a)^{1/2}} \cdot \frac{1}{q+k} \cdot e^{qx-qa-kb}$$

$$-I_1(qx)k_1(qa)k_1(qb)V_2(b) \cong \frac{A_1}{4q^2} \cdot \frac{1}{(x.a)^{1/2}} \cdot \frac{1}{q-k} \cdot e^{qx-qa-kb}$$

$$k_1(qb)k_1(qa)V_2(a)I_1(qx) \cong -\frac{A_1}{4q^2} \cdot \frac{1}{(x.b)^{1/2}} \cdot \frac{1}{q-k} \cdot e^{qx-qb-ka}$$

$$k_1(qb)I_1(qa)V_1(a)I_1(qx) \cong -\frac{A_1}{4q^2} \cdot \frac{1}{(x.b)^{1/2}} \cdot \frac{1}{q+k} \cdot e^{qx-qb-ka}$$

$$-I_1(qb)I_1(qa\ V_1(a)k_1(qx) \cong \frac{A_1}{4q^2} \cdot \frac{1}{(x.b)^{1/2}} \cdot \frac{1}{q+k} \cdot e^{qb-qx-ka}$$

$$-k_1(qx)I_1(qb)k_1(qa)V_2(a) \cong \frac{A_1}{4q^2} \cdot \frac{1}{(x.b)^{1/2}} \cdot \frac{1}{q-k} \cdot e^{qb-qx-ka}$$

$$I_1(qa)I_1(qb)V_1(b)k_1(qx) \cong -\frac{A_1}{4q^2} \cdot \frac{1}{(x.a)^{1/2}} \cdot \frac{1}{q+k} \cdot e^{qa-qx-kb}$$

$${}_1(qb)V_2(b) \cong -\frac{A_1}{4q^2} \cdot \frac{1}{(x.a)^{1/2}} \cdot \frac{1}{q-k} \cdot e^{qa-qx-kb} \qquad \text{...(18)}$$

$$-\frac{P}{p+\omega_1}\left[I_1(qb)k_1(qx)-k_1(qb)I_1(qx)\right]\cong-\frac{P}{p+\omega_1}\cdot\frac{e^{q(b-x)}}{2q(bx)^{1/2}}$$

$$\left[1+\frac{3}{8q}\cdot\frac{b-x}{bx}-e^{-2q(b-x)}\cdot\left(1-\frac{b-x}{bx}\cdot\frac{3}{8q}\right)\right]$$

(Sarkar 1967)...(19)

Similarly (Sarkar 1967)

$$k_1(qa)I_1(qb)-k_1(qb)I_1(qa)\cong\frac{e^{q(b-a)}}{2q(ba)^{1/2}}$$

$$\left[1+\frac{3}{8q}\cdot\frac{b-a}{ba}-e^{-2q(b-a)}\left\{1-\frac{(b-a)}{ba}\cdot\frac{3}{8q}\right\}\right] \quad ...(20)$$

Hence, neglecting the terms containing $\frac{1}{q^2}$ and its higher powers from (18), (19) and (20) we get

$$-\frac{P}{p+\omega_1}\cdot\frac{[I_1(qb)k_1(qx)-k_1(qb)I_1(qx)]}{[k_1(qa)I_1(qb)-k_1(qb)I_1(qa)]}\cong-\frac{P}{P+\omega_1}\cdot$$

$$\left(\frac{a}{x}\right)^{1/2}\cdot e^{q(a-x)}\cdot\left[1+\frac{1}{q}\left\{\frac{3(b-x)}{8bx}-\frac{3(b-a)}{8ab}\right\}\right]$$

$$\cong-\frac{P}{p+\omega_1}\left(\frac{a}{x}\right)^{1/2}\cdot e^{q(a-x)}, \quad ...(21)$$

and

$$I_1(qx)V_1(x)+k_1(qx)V_2(x)\cong-A_1\cdot\frac{e^{-kx}}{(2k)}\left[\frac{1}{q-k}-\frac{1}{q+k}\right] \quad ...(22)$$

From equations (12), (14), (21) and (22) we get

$$\bar{\sigma}\cong-P\cdot\left(\frac{a}{x}\right)^{1/2}\cdot\frac{1}{p+\omega_1}\cdot e^{q(a-x)}-A_1e^{-kx}\cdot\frac{1}{(2k)}\left[\frac{1}{q-k}-\frac{1}{q+k}\right]$$

...(23)

Putting $a-x=-h_1$ in (23) we get

$$\bar{\sigma} \quad -P\cdot\left(\frac{a}{x}\right)^{1/2}\left[\frac{1}{p+\omega_1}\cdot e^{-h_1q}\right]-\frac{A_1}{2k}\cdot e^{-kx}$$

$$\left[\frac{1}{q-k}-\frac{1}{q+k}\right] \quad ...(24)$$

Now, putting the value of $q^2=\frac{(p^2+Rp)^2}{c^2(p+S)^2}$ we have

$$\frac{1}{q-k}=\frac{C(p+S)}{\left(p+\frac{R-kc}{2}\right)^2-N^2}$$

and

$$\frac{1}{q+K}=\frac{c(p+S)}{\left(p+\frac{R+kc}{2}\right)^2-N_1^2}$$

...(25)

where $N^2 = \left(\frac{R - kc}{2} \right)^2 + kcS$ and $N_1^2 = \left(\frac{R + kc}{2} \right)^2 - kcS$

Using the notation of Van der Pol & Bremmer in (25) we have

$$x(t).\, e^{c_3 t} \doteqdot \frac{c}{2N} \left[\frac{N - S_1}{p + N} + \frac{N + S_1}{p - N} \right]$$

and $$x_1(t).\, e^{c_4 t} \doteqdot \frac{c}{2N_1} \left[\frac{N_1 - S_2}{p + N_1} + \frac{N_1 + S_2}{p - N_1} \right] \quad \text{...(26)}$$

where $c_3 = \frac{R - kc}{2}$, $c_4 = \frac{R + kc}{2}$, $S_1 = S - \frac{R - kc}{2}$,

$$S_2 = S - \frac{R + kc}{2}.$$

Again, using the value of q^2 we have

$$\frac{1}{p + \omega_1} .\, e^{-h_1 q}\, \frac{e^{-\frac{h_1}{c}[p - (S - R)]}}{p + \omega_1} - \frac{e^{-\frac{h_1}{c}[p - (S - R)]}}{p + \omega_1} . \left[1 + e^{-\frac{c_5}{p + S}} \right] \quad \text{...(27)}$$

where $c_5 = \frac{h_1}{c} S(S - R)$.

Using the notation of Van der Pol & Bremmer, we write

$$x_2(t) \doteqdot \frac{e^{-\frac{h_1}{c}[p - (S - R)]}}{p + \omega_1} \quad \text{and} \quad x_3(t) \doteqdot \left[1 - e^{-\frac{c_5}{p + S}} \right]$$

$$\left.\begin{aligned} \text{Hence,} \quad x_2(t) .\, e^{(R - S)} &= 0, \quad 0 < t < \frac{h_1}{c} \\ &= e^{-\alpha\left(t - \frac{h_1}{c}\right)}, \quad t > \frac{h_1}{c} \end{aligned}\right\} \quad \text{(Erdelyi)} \quad \text{.. (28)}$$

where $\alpha = \omega_1 + S - R$,

and $$x_3(t) .\, e^{St} = c_5^{1/2} .\, t^{-1/2} .\, J_1(2 c_5^{1/2} t^{1/2}) \quad \text{(Erdelyi)} \quad \text{...(29)}$$

Using (24), (26), (28) and (29) we get

$$\sigma \cong -e^{-kx} \left[Ee^{-\xi_1 t} + Fe^{\xi_2 t} - Ge^{-\xi_3 t} - He^{-\xi_4 t} \right], \quad 0 < t < \frac{h_1}{c} \quad \text{...(30)}$$

and
$$\sigma \cong -P\left(\frac{a}{x}\right)^{1/2}\Big[e^{-\omega_1 t} \cdot e^{\frac{h_1 \propto}{c}} - e^{-\omega_1 t} \cdot e^{\frac{h_1 \propto}{c}} * c_5^{1/2} \cdot e^{-St} \cdot t^{-1/2} \cdot J_1\left(2c_5^{1/2}\, t^{1/2}\right)\Big]$$
$$- e^{-kx}\left[E\, e^{-\xi_1 t} + F\, e^{\xi_2 t} - G\, e^{-\xi_3 t} - H\, e^{-\xi_4 t}\right], \quad t > \frac{h_1}{c}$$
$$= -P\left(\frac{a}{x}\right)^{1/2} \cdot e^{\frac{h_1 \propto}{c}} \cdot e^{-\omega_1 t}\Big[1 + c_5^{1/2} \int_0^t e^{(\omega_1 - S)\tau} \cdot \tau^{-1/2} \cdot J_1\left(2c_5^{1/2}\, \tau^{1/2}\right) d\tau\Big]$$
$$- e^{-kx}\left(E\, e^{-\xi_1 t} + F e^{\xi_2 t} - G e^{-\xi_3 t} - H e^{-\xi_4 t}\right], \quad t > \frac{h_1}{c}.$$

(Churchill 1958) ..(31)

where $E = \frac{A_1}{2k} \cdot \frac{c}{2N}(N - S_1)$, $F = \frac{A_1}{2K} \cdot \frac{c}{2N}(N + S_1)$,

$G = \frac{A_1}{2k} \cdot \frac{c}{2N_1}(N_1 - S_2)$

$H = \frac{A_1}{2k} \cdot \frac{c}{2N_1} \cdot (N_1 + S_2)$, $\xi_1 = (c_3 + N)$,

$\xi_2 = (N - c_3)$, $\xi_3 = (c_4 + N_1)$, $\xi_4 = (c_4 - N_1)$.

Now, $J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{1+2r}}{r!\, \Gamma(r+2)}$, $e^{Q\tau} = \sum_{n=0}^{\infty} \frac{(Q\tau)^n}{n!}$

$$\left.\begin{array}{ll} & \text{if } \omega_1 - S = Q > 0 \\ \text{and } e^{-Q\tau} = \sum_{n=0}^{\infty} \frac{(-1)^n (Q\tau)^n}{n!} & \text{if } \omega_1 - S = -Q < 0 \end{array}\right\} \quad \dots(32)$$

Therefore, using (32) we get

$$\left.\begin{array}{l} \int_0^t e^{(\omega_1 - S)\tau} \cdot \tau^{-1/2} \cdot J_1\left(2c_5^{1/2}\, \tau^{1/2}\right) d\tau \\ = \sum_{r,n=0}^{\infty} \frac{(-1)^r \cdot c_5^{\frac{1+2r}{2}} \cdot Q^n}{r!\, n!\, \Gamma(r+2) \cdot (r+n+1)} \cdot t^{r+n+1}, \text{ if } \omega_1 - S > 0 \\ \text{and} \\ \int_0^t e^{(\omega_1 - S)\tau} \cdot \tau^{-1/2} \cdot J_1\left(2c_5^{1/2}\, \tau^{1/2}\right) d\tau \\ = \sum^{\infty} \frac{(-1)^{r+n} \cdot c_5^{\frac{1+2r}{2}} \cdot Q^n}{r!\, n!\, \Gamma(r+2)(r+n+1)} \cdot t^{r+n+1}, \text{ if } \omega_1 - S < 0 \end{array}\right\} (33)$$

Substituting the value of h_1 in (29), (31) and using (33) we get

$$\sigma \cong -e^{-kx}\left[Ee^{-\zeta_1 t}+Fe^{-\zeta_2 t}-Ge^{-\zeta_3 t}-He^{-\zeta_4 t}\right],\quad 0<t<\frac{x-a}{c} \qquad \ldots(34)$$

$$\sigma \cong -P\left(\frac{a}{x}\right)^{1/2}.\ e^{\left(\frac{x-a}{c}\right)\alpha}.\ e^{-\omega_1 t}\left[1-\right.$$

$$\left. c_5\sum_{r,n=0}^{\infty}\frac{(-1)^r.\ c_5{}^r.\ Q^n}{r\,!\,n\,!\,(r+n+1)\Gamma(r+2)},\ t^{r+n+1}\right]$$

$$-e^{-kx}\left[Ee^{-\zeta_1 t}+Fe^{\zeta_2 t}-Ge^{-\zeta_3 t}-He^{-\zeta_4 t}\right],$$

$$\omega_1-S<0 \text{ and } t>\frac{x-a}{c}$$

$$\sigma \cong -P\left(\frac{a}{x}\right)^{1/2}.\ e^{\left(\frac{x-a}{c}\right)\alpha}.\ e^{-\omega_1 t}\left[1-\right.$$

$$\left. c_5\sum_{r,n=0}^{\infty}\frac{(-1)^{r+n}.\ c_5{}^r.\ Q^n}{r\,!\,n\,!\,(r+n+1)\Gamma(r+2)},\ t^{r+n+1}\right]$$

$$+e^{-kx}\left[Ee^{-\zeta_1 t}+Fe^{\zeta_2 t}-Ge^{-\zeta_3 t}-He^{-\zeta_4 t}\right],$$

$$\omega_1-S<0 \text{ and } t>\frac{x-a}{c}. \qquad \ldots(35)$$

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References

Churchill, R. V. 1958 *Operational Mathematics*, 2nd ed.

Erdelyi—*Tables of Integral Transforms*, 242 and 245.

Inche E. L. 1956 *Ordinary differential equation—Dover Publication, Inc.* 122

Sarkar, S. K. 1967 Note on the vibration of visco-elastic rod of variable cross section—*Tomul. XIII (XVII), Fasc.* 1-2,

Bulletin of the Calcutta Mathematical Society 1965, 57